# Section 12.6

## A Survey of Quadric Surfaces



PreLecture Review Video


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by Joseph Phillip Brennan
Jila Niknejad

# 1 Conic Sections

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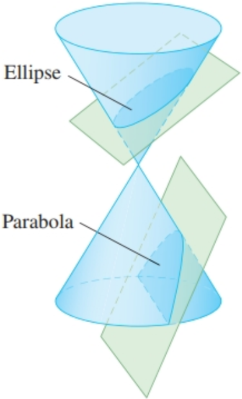
by Joseph Phillip Brennan
Jila Niknejad

# Review: Conic Sections

In $\mathbb{R}^2$, **conic sections** are curves satisfying an equation of the form

$$Ax^2 + Bx + Cy^2 + Dy + E = 0.$$




## Ellipse

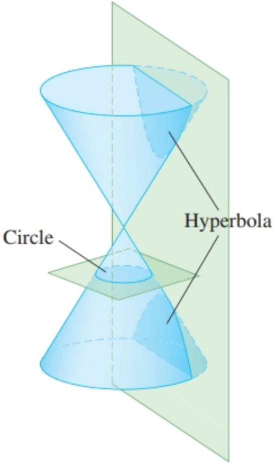
$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

## Hyperbola

$$x^2 - y^2 = a^2$$

$$y^2 - x^2 = a^2$$

Video



# 2 Quadric Surfaces

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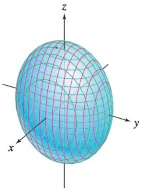
Jila Niknejad

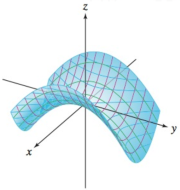
# Quadric Surfaces

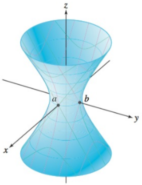
In $\mathbb{R}^3$, a **quadric surface** is a surface satisfying an equation of the form

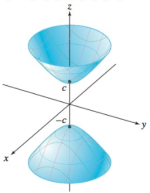
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + ax + by + cz + d = 0.$$

In this course we discuss quadric surfaces in standard form. That is, $D, E$ and $F$ are all zero and at least one of $A, B$ and $C$ is not zero. A quadric surface is classified by the curves obtained by intersecting it with planes parallel to the $xy$, $xz$, and $yz$ planes. (These intersection curves will often be conic sections.)









# Quadric Surfaces

There are many kinds of quadric surfaces (more than there are conics):

*conic cylinders; ellipsoids (including spheres); elliptic paraboloids; hyperbolic paraboloids; hyperboloids of one sheet; hyperboloids of two sheets; cones*

- What is important is knowing how to recognize a quadric surface from its equation or its cross-section curves, or vice versa. (The names aren't too important.)
- The cross-section curves are all conic sections (defined by degree-2 polynomials).
- Cross-section Curves are the intersection of the surface and a plane; the resulting curve on the plane is a two dimensional curves in three dimensional space; also denoted by the trace of the surface on the plane.
- In this section, to find a cross-section curve, **set one of the variables equal to a constant**.

# 3 Conic Cylinder

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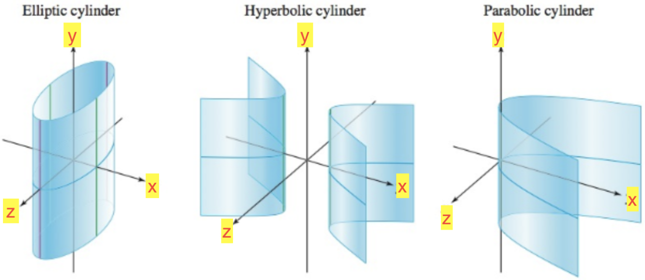
Jila Niknejad

# Conic Cylinder

Surfaces with an equation containing only 2 of the three variables. Cylinders can "slide" along the axis of the **missing variable**. That is, for all values of the missing variable, the cross-section curves are the same. Link




$$x^2 + \frac{z^2}{9} = 1$$

Cross-section Curves: lines and ellipses.

$$x^2 - \frac{z^2}{9} = 1$$

Cross-section Curves: lines and hyperbolas.

$$x = z^2$$

Cross-section curves: lines and parabolas.

$y$ **is a free variable.**

# 4 Ellipsoid

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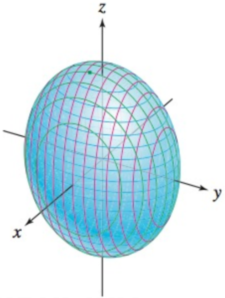
Jila Niknejad

# Ellipsoid

Cross-sectionals are ellipses or points.

$$\left(\frac{\boldsymbol{x}}{\boldsymbol{a}}\right)^2 + \left(\frac{\boldsymbol{y}}{\boldsymbol{b}}\right)^2 + \left(\frac{\boldsymbol{z}}{\boldsymbol{c}}\right)^2 = 1$$




For example, intersecting the ellipsoid

$$\left(\frac{x}{2}\right)^2 + y^2 + \left(\frac{z}{3}\right)^2 = 1$$

with the plane $z = 1$ gives the ellipse

$$\left(\frac{x}{2}\right)^2 + y^2 + \left(\frac{1}{3}\right)^2 = 1$$

or

$$\left(\frac{x}{2}\right)^2 + y^2 = \frac{8}{9}$$

$a, b, c$ = radii in $x$, $y$, $z$ directions

# 5 Elliptic Parabloid

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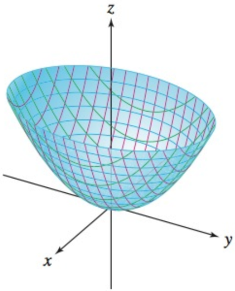
# Elliptic Paraboloid




$$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

- Cross-sectionals:

| Plane | Cross-section curves |
|---|---|
| $z = k$ | Ellipses or a point |

| Plane | Cross-section curves |
|---|---|
| $x = k$ or $y = k$ | Parabolas |

- Surface passes the Vertical Line Test (since $z$ is a function of $x$ and $y$).
- When $a = b$, the horizontal intersections are circles.

| | | |
|---|---|---|
| Intersecting with $z = k$:constant $> 0$ is an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$. | Intersecting with $z = k$:constant $< 0$ is not Possible. | Intersecting with $z = 0$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, a point: $(0,0,0)$. |

| | |
|---|---|
| Intersecting with $x = k$:constant, is a parabola: $z = \frac{k^2}{a^2} + \frac{y^2}{b^2}$ | Intersecting with $y = k$:constant, is a parabola: $z = \frac{x^2}{a^2} + \frac{k^2}{b^2}$ |

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# 6 Hyperbolic Paraboloid

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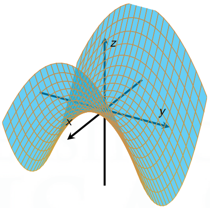
# Hyperbolic Paraboloid (Saddle Surface)

$$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

| Plane | Cross-section curves |
|---|---|
| $x = k$ | Parabolas (downward) |
| $y = k$ | Parabolas (upward) |
| $z = k$ | Hyperbolas |
| $z = 0$ | Two intersecting lines |




(passes Vert. Line Test)

Link

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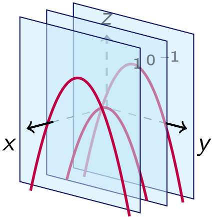
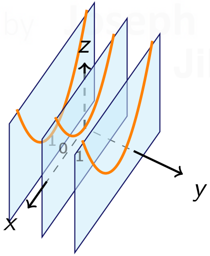
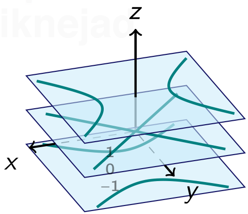
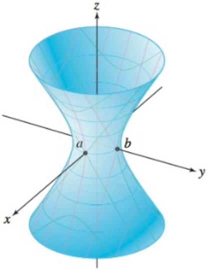


# Hyperboloid of One Sheet

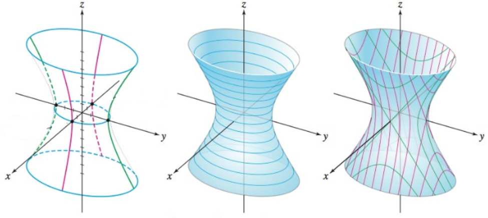
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$$




Cross-sections:

| Plane | Cross-section curves |
|---|---|
| $z = k$ | Ellipses |

| Plane | Cross-section curves |
|---|---|
| $x = k \neq a$ or $y = k \neq b$ | Hyperbolas |



# Cone

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$$

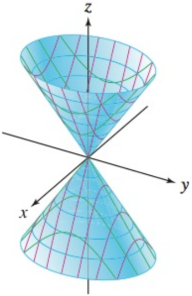
- Cross-sectionals:

| Plane | Cross-sectionals |
|---|---|
| $z = k$ | Ellipses or a point |

| Plane | Cross-section curves |
|---|---|
| $x = k$ or $y = k$ | Hyperbolas or pairs of lines |

  Intersecting with $z = 0$ results in a single point. Intersecting with $x = 0$ or $y = 0$ results in a pair of lines.
- Halves of the cone are called *nappes*.
- Nappes are the graphs of the functions

$$z = \pm c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}.$$



# 7 Hyperboloid of Two Sheets

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by Joseph Phillip Brennan
Jila Niknejad

# Hyperboloid of Two Sheets

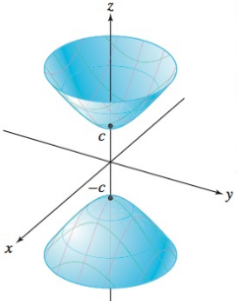
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$$

- Cross-sectionals:

| Plane | Cross-sectionals |
|---|---|
| $z = k$ and $\lvert k\rvert \geq \lvert c\rvert$ | Ellipses or a point |

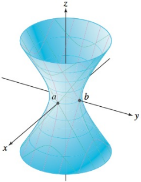
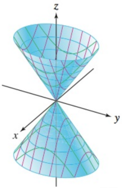
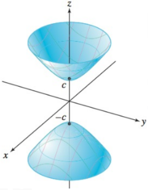
| Plane | Cross-section curves |
|---|---|
| $x = k$ or $y = k$ | Hyperbolas |

- If $-\lvert c\rvert < z < \lvert c\rvert$, there are no $(x, y)$ solutions to the equation (because then the RHS is negative and the LHS has to be positive).
- All solutions have $z \geq \lvert c\rvert$ (top sheet) or $z \leq -\lvert c\rvert$ (bottom sheet).
- Each sheet is the graph of a function.



# Hyperboloids and Cones — Comparison

What do the equations have in common? When all variables are moved to one side of the equation, all three variables appear in power two; one of the three has a different sign than the others, we use the intercept(s) of that variable to quickly identify the surface.

| Hyperboloid of 1 sheet | Cone | Hyperboloid of 2 sheets |
| --- | --- | --- |
|  |  |  |
| $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ |
| No $z$-intercept. All $z$-cross sections are possible. | One $z$-intercept. All $z$-cross sections are possible. | Two $z$-intercepts. Some $z$-cross Sections are not possible. |

# 8 Comparing Quadric Surfaces

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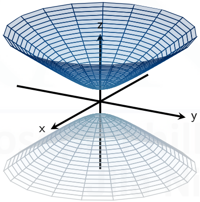
Jila Niknejad

# Comparing Some Quadrics




$$z^2 = x^2 + y^2 + a \text{ for } a = 4.$$

Link

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# 9 More Examples

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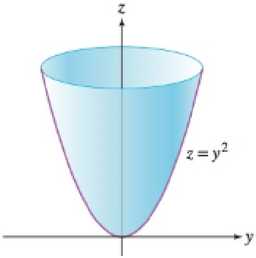
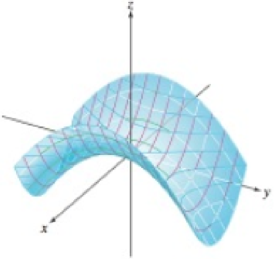
**Example 1:** Sketch and identify the graphs of

| | (a) $z = 3x^2 + 4y^2$ | (b) $z = 2x^2 - 5y^2$ |
|---|---|---|
| Passes VLT? | Yes | Yes |
| Intersect with $x = k$ | Parabola (up) | Parabola (down) |
| Intersect with $y = k$ | Parabola (up) | Parabola (up) |
| Intersect with $z = k$ | Ellipse or a Point | Hyperbola or Two Lines |




(a) Elliptic paraboloid




(b) Hyperbolic paraboloid

▸ Video

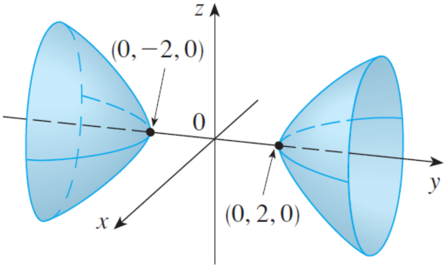
**Example 2:** Sketch and identify the graph of

$$4x^2 - y^2 + 9z^2 + 4 = 0$$

Solution: Rewrite the equation as $4x^2 + 9z^2 = y^2 - 4$ or equivalently

$$x^2 + \left(\frac{3z}{2}\right)^2 - \left(\frac{y}{2}\right)^2 = -1.$$

| | |
|---|---|
| Intersect with $x = k$ | Hyperbola |
| Intersect with $y = k$ (if $\lvert k\rvert > 2$) | Ellipse |
| Intersect with $y = k$ (if $\lvert k\rvert < 2$) | Nothing! |
| Intersect with $z = k$ | Hyperbola |




Hyperboloid of two sheets

Video

Summary Video

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